Section 2
Dimensional Analysis

2.1 General. Dimensional analysis is essentially a means of utilizing a partial knowledge of a problem when the details are too obscure to permit an exact analysis. See Taylor, E. S. (1974). It has the enormous advantage of requiring for its application a knowledge only of the variables which govern the result. To apply it to the flow around ships and the corresponding resistance, it is necessary to know only upon what variables the latter depends. This makes it a powerful tool, because the correctness of a dimensional solution does not depend upon the soundness of detailed analyses, but only upon the choice of the basic variables. Dimensional solutions do not yield numerical answers, but they provide the form of the answer so that every experiment can be used to the fullest advantage in determining a general empirical solution.

2.2 Dimensional Homogeneity. Dimensional analysis rests on the basic principle that every equation which expresses a physical relationship must be dimensionally homogeneous. There are three basic quantities in mechanics—mass, length and time—which are represented by the symbols M, L, and T. Other quantities, such as force, density, and pressure, have dimensions made up from these three basic ones.

Velocity is found by dividing a length or distance by a time, and so has the dimensions L/T. Acceleration, which is the change in velocity in a certain time, thus has dimensions of $(L/T)/T$, or L/T^2.

Force, which is the product of mass and acceleration, has dimensions of $M \times L/T^2$ or ML/T^2.

As a simple case to illustrate the principle of dimensional analysis, suppose we wish to determine an expression for the time of swing of a simple pendulum.

If T is the period of such a pendulum in vacuo (so that there is no frictional damping), it could depend upon certain physical quantities such as the mass of the pendulum bob, m, the length of the cord, l, (supposed to be weightless) and the arc of swing, s. The force which operates to restore the pendulum to its original position when it is disturbed is its weight, mg, and so the acceleration due to gravity, g, must be involved in the problem.

We can write this in symbols as

$$T = f(m, l, s, g)$$

where f is a symbol meaning "is some function of."

If we assume that this function takes the form of a power law, then

$$T = m^a \, l^b \, s^c \, g^d$$

If this equation is to fulfill the principle of dimensional homogeneity, then the dimensions on each side must be the same. Since the left-hand side has the dimension of time only, so must the right-hand side.

Writing the variables in terms of the fundamental units, we have

$$T^1 = M^a L^b L^c \, (L/T^2)^d$$

Equating the exponents of each unit from each side of the equation, we have

$$\begin{aligned} a &= 0 \\ b + c + d &= 0 \\ -2d &= 1 \end{aligned}$$

Hence

$$\begin{aligned} d &= -1/2 \\ a &= 0 \\ b + c &= 1/2 \end{aligned}$$

The expression for the period of oscillation T seconds is therefore

$$T = \text{constant} \times l^{1/2-c} \times s^c \times g^{-1/2}$$

$$= \text{constant} \times \sqrt{l/g} \times (s/l)^c$$

The solution indicates that the period does not depend on the mass of the bob, but only on the length, the acceleration due to gravity, and the ratio of length of arc to length of pendulum. The principle of dimensions does not supply the constant of proportionality, which must be determined experimentally.

The term (s/l) is a mere number, each quantity being of dimension L, and dimensionally there is no restriction on the value of c. We can therefore write

$$\mathrm{T} = \text{constant} \times \sqrt{l/g} \times f(s/l) \qquad (2)$$

Although the form of the function f is undetermined, it is explicitly indicated by this equation that it is not the arc s itself which is important, but its ratio to l: i.e., the maximum angle of swing, s/l radians.

The function f can be found by experiment, and must approach the value unity for small swings, so as to lead to the usual formula for a simple pendulum under such conditions:

$$T = \text{constant} \times \sqrt{l/g}$$

The most important question regarding any dimensional solution is whether or not physical reasoning has led to a proper selection of the variables which govern the result.

Applying dimensional analysis to the ship resistance problem, the resistance R could depend upon the following:

(*a*) Speed, V.
(*b*) Size of body, which may be represented by the linear dimension, L.
(*c*) Mass density of fluid, ρ (mass per unit volume)
(*d*) Viscosity of fluid, μ
(*e*) Acceleration due to gravity, g
(*f*) Pressure per unit area in fluid, p

It is assumed that the resistance R can now be written in terms of unknown powers of these variables:

$$R \propto \rho^a V^b L^c \mu^d g^e p^f \qquad (3)$$

Since R is a force, or a product of mass times acceleration, its dimensions are ML/T^2.

The density ρ is expressed as mass per unit volume, or M/L^3.

In a viscous fluid in motion the force between adjacent layers depends upon the area A in contact, the coefficient of viscosity of the liquid and upon the rate at which one layer of fluid is moving relative to the next one. If u is the velocity at a distance y from the boundary of the fluid, this rate or velocity gradient is given by the expression du/dy.

The total force is thus

$$F = \mu A du/dy$$

du/dy being a velocity divided by a distance has dimensions of $(L/T)/L$, *or* $1/T$, and the dimensional equation becomes

$$ML/T^2 = \mu L^2 \times 1/T$$

or

$$\mu = M/LT$$

p is a force per unit area, and its dimensions are $(ML/T^2)/L^2$, or M/LT^2.

The ratio μ/ρ is called the kinematic viscosity of the liquid, ν, and has dimensions given by

$$\nu = \mu/\rho = (M/LT)\cdot(L^3/M) = L^2/T$$

Introducing these dimensional quantities into Equation (3), we have

$$ML/T^2 = (M/L^3)^a (L/T)^b (L)^c (M/LT)^d \times (L/T^2)^e (M/LT^2)^f \quad (4)$$

whence

$$\left.\begin{aligned} a + d + f &= 1 \\ -3a + b + c - d + e - f &= 1 \\ b + d + 2e + 2f &= 2 \end{aligned}\right\}$$

or

$$\left.\begin{aligned} a &= 1 - d - f \\ b &= 2 - d - 2e - 2f \end{aligned}\right\}$$

and

$$\begin{aligned} c &= 1 + 3a - b + d - e + f \\ &= 1 + 3 - 3d - 3f - 2 + d \\ &\quad + 2e + 2f + d - e + f \\ &= 2 - d + e \end{aligned}$$

Then from Equation (3)

$$R \propto \rho V^2 L^2 f \left[\left(\frac{\rho V L}{\mu}\right)^{-d} \left(\frac{gL}{V^2}\right)^{e} \left(\frac{p}{\rho V^2}\right)^{f} \right] \quad (5)$$

All three expressions within the brackets are nondimensional, and are similar in this respect to the s/l term in Equation (2). There is therefore no restriction dimensionally on the exponents d, e, and f. The form of the function f must be found by experiment, and may be different for each of the three terms.

Writing ν for μ/ρ and remembering that for similar shapes the wetted surface S is proportional to L^2, Equation (5) may be written

$$\frac{R}{\frac{1}{2}\rho S V^2} = f\left[\frac{VL}{\nu}, \frac{gL}{V^2}, \frac{p}{\rho V^2}\right] \quad (6)$$

where the left-hand side of the equation is a nondimensional resistance coefficient. Generally in this chapter R will be given in kN and ρ in kg/L (or t/m^3), although N and kg/m^3 are often used (as here) in the cases of model resistance and ship air/wind resistance.

Equation (6) states in effect that if all the parameters on the right-hand side have the same values for two geometrically similar but different sized bodies, the flow patterns will be similar and the value of $R/\frac{1}{2}\rho S V^2$ will be the same for each.

2.3 Corresponding Speeds. Equation (6) showed how the total resistance of a ship depends upon the various physical quantities involved, and that these are associated in three groups, VL/ν, gL/V^2 and $p/\rho V^2$.

Considering first the case of a nonviscous liquid in which there is no frictional or other viscous drag, and neglecting for the moment the last group, there is left the parameter gL/V^2 controlling the surface wave system, which depends on gravity. Writing the wave-making or residuary resistance as R_R and the corresponding coefficient as C_R, C_R can be expressed as

$$C_R = \frac{R_R}{\frac{1}{2}\rho S V^2} = f_1(V^2/gL) \quad (7)$$

This means that *geosims*[3] (geometrically similar bodies) of different sizes will have the same specific residuary resistance coefficient C_R if they are moving at the same value of the parameter V^2/gL.

According to Froude's *Law of Comparison*[4]: "The (residuary) resistance of geometrically similar ships is in the ratio of the cube of their linear dimensions if their speeds are in the ratio of the square roots of their linear dimensions." Such speeds he called *corresponding speeds.*[5] It will be noted that these corresponding speeds require $V/\sqrt{L}$ to be the same for model and ship, which is the same condition as expressed in Equation (7). The ratio $V_K/\sqrt{L}$, commonly with V_k in knots and L in feet, is called the speed-length ratio. This ratio is often used in presenting resistance data because of the ease of evaluating it arithmetically, but it has the drawback of not being nondimensional. The value of $V/\sqrt{gL}$, on the other hand, is nondimensional and has the same numerical value in any consistent system of units. Because of Froude's close association with the concept of speed-length ratio, the parameter $V/\sqrt{gL}$ is called the Froude number, with the symbol Fn.

When V_k is expressed in knots, L in feet, and g in ft/sec^2, the relation between $V/\sqrt{L}$ and Froude number is

$$\text{Fn} = 0.298\, V_k/\sqrt{L}$$

or

$$V_k/\sqrt{L} = 3.355\text{Fn}$$

[3] A term first suggested by Dr. E.V. Telfer.

[4] Stated in 1868 by William Froude (1955) who first recognized the practical necessity of separating the total resistance into components, based on the general law of mechanical similitude, from observations of the wave patterns of models of the same form but of different sizes.

The residuary resistances of ship (R_{RS}) and of model (R_{RM}) from Equation (7) will be in the ratio

$$\frac{R_{RS}}{R_{RM}} = \frac{\frac{1}{2}\rho S_S V_S^2 C_{RS}}{\frac{1}{2}\rho S_M V_M^2 C_{RN}}$$

where subscripts S and M refer to ship and model, respectively.

If both model and ship are run in water of the same density and at the same value of V^2/gL, as required by Equation (7), i.e.

$$(V_S)^2/gL_S = (V_M)^2/gL_M$$

then C_R will be the same for each, and

$$R_{RS}/R_{RM} = S_S\,(V_S)^2/S_M\,(V_M)^2 = (L_S)^2/(L_M)^2\;L_S/L_M$$
$$= (L_S)^3/(L_M)^3 = \Delta_S/\Delta_M \quad (8)$$

where Δ_S and Δ_M are the displacements of ship and model, respectively.

This is in agreement with Froude's law of comparison.

It should be noted from Equation (8) that at corresponding speeds, i.e., at the same value of $V/\sqrt{L}$

$$R_{RS}/\Delta_S = R_{RM}/\Delta_M \quad (9)$$

i.e., the residuary resistance per unit of displacement is the same for model and ship. Taylor made use of this in presenting his contours of residuary resistance in terms of pounds resistance per long ton of displacement (Section 8.6).

If the linear scale ratio of ship to model is λ, then the following relations hold:

$$L_S/L_M = \lambda$$
$$V_S/V_M = \sqrt{L_S}/\sqrt{L_M} = \sqrt{\lambda} = \lambda^{1/2} \quad (10)$$
$$R_{RS}/R_{RM} = (L_S)^3/(L_M)^3 = \Delta_S/\Delta_M = \lambda^3$$

The "corresponding speed" for a small model is much lower than that of the parent ship. In the case of a 5 m model of a 125 m ship (linear scale ratio $\lambda = 25$), the model speed corresponding to 25 knots for the ship is $25/\lambda^{1/2}$, or $25/\sqrt{25}$, or 5 knots. This is a singularly fortunate circumstance, since it enables ship models to be built to reasonable scales and run at speeds which are easily attainable in the basin.

Returning to Equation (6), consider the last term, $p/\rho V^2$. If the atmospheric pressure above the water surface is ignored and p refers only to the water head, then for corresponding points in model and ship p will vary directly with the linear scale ratio λ. At corresponding speeds V^2 varies with λ in the same way so that $p/\rho V^2$ will be the same for model and ship. Since the atmospheric pressure is usually the same in model and ship, when it is included in p, so that the latter is the *total* pressure at a given point, the value of $p/\rho V^2$ will be much greater for model than for ship. Fortunately, most of the hydrodynamic forces arise from differences in *local* pressures, and these are proportional to V^2, so that the forces are not affected by the atmospheric pressure so long as the fluid remains in contact with the model and ship surfaces. When the pressure approaches very low values, however, the water is unable to follow surfaces where there is some curvature and cavities form in the water, giving rise to *cavitation*. The similarity conditions are then no longer fulfilled. Since the absolute or total pressure is greater in the model than in the ship, the former gives no warning of such behavior. For tests in which this danger is known to be present, special facilities have been devised, such as variable-pressure water tunnels, channels or towing basins, where the correctly scaled-down total pressure can be attained at the same time that the Froude condition is met.

In the case of a deeply submerged body, where there is no wavemaking, the first term in Equation (6) governs the frictional resistance, R_F. The frictional resistance coefficient is then

$$C_F = \frac{R_F}{\frac{1}{2}\rho S V^2} = f_2(VL/\nu) \quad (11)$$

and C_F will be the same for model and ship provided that the parameter VL/ν is the same. This follows essentially from the work of Osborne Reynolds (1883), for which reason the product VL/ν is known as Reynolds number, with the symbol Rn.

If both model and ship are run in water at the same density and temperature, so that ν has the same value, it follows from (11) that $V_S L_S = V_M L_M$. This condition is quite different from the requirement for wave-making resistance similarity. As the model is made smaller, the speed of test must increase. In the case already used as an illustration, the 5-m model of a 125-m, 25-knot ship would have to be run at a speed of 625 knots.

The conditions of mechanical similitude for both friction and wave-making cannot be satisfied in a single test. It might be possible to overcome this difficulty by running the model in some other fluid than water, so that the change in value of ν would take account of the differences in the VL product. In the foregoing example, in order to run the model at the correct wave-making corresponding speed, and yet keep the value of VL/ν the same for both model and ship, a fluid would have to be found for use with the model which had a kinematic viscosity coefficient only 1/125 that of water. No such fluid is known. In wind-tunnel work, similitude can be attained by using compressed air in the model tests, so decreasing ν and increasing VL/ν to the required value.

The practical method of overcoming this fundamental difficulty in the use of ship models is to deal with

[5] This same law had previously been put forward by the French Naval Constructor Reech in 1832, but he had not pursued it or demonstrated how it could be applied to the practical problem of predicting ship resistance (Reech, 1852).

the frictional and the wave-making resistances separately, by writing

$$C_T = C_R + C_F \tag{12}$$

This is equivalent to expressing Equation (6) in the form

$$C_T = \frac{R_T}{\frac{1}{2}\rho S V^2} = f_1(V^2/gL) + f_2(VL/\nu) \tag{13}$$

Froude recognized this necessity, and so made ship-model testing a practical tool. He realized that the frictional and residuary resistances do not obey the same law, although he was unaware of the relationship expressed by Equation (11).

2.4 Extension of Model Results to Ship To extend the model results to the ship, Froude proposed the following method, which is based on Equation (12). Since the method is fundamental to the use of models for predicting ship resistance, it must be stated at length:

Froude noted:

(*a*) The model is made to a linear scale ratio of λ and run over a range of "corresponding" speeds such that $V_S/\sqrt{L_S} = V_M/\sqrt{L_M}$.

(*b*) The total model resistance is measured, equal to R_{TM}.

(*c*) The frictional resistance of the model R_{FM} is calculated, assuming the resistance to be the same as that of a smooth flat plank of the same length and surface as the model.

(*d*) The residuary resistance of the model R_{RM} is found by subtraction:

$$R_{RM} = R_{TM} - R_{FM}$$

(*e*) The residuary resistance of the ship R_{RS}, is calculated by the law of comparison, Equation (10):

$$R_{RS} = R_{RM} \times \lambda^3$$

This applies to the ship at the corresponding speed given by the expression

$$V_S = V_M \times \lambda^{1/2}$$

(*f*) The frictional resistance of the ship R_{FS} is calculated on the same assumption as in footnote (4), using a frictional coefficient appropriate to the ship length.

(*g*) The total ship resistance (smooth hull) R_{TS} is then given by

$$R_{TS} = R_{FS} + R_{RS}$$

This principle of extrapolation from model to ship is still used in all towing tanks, with certain refinements to be discussed subsequently.

Each component of resistance will now be dealt with in greater detail.